# The Classification of Finite Coxeter Groups

Zero

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# Outline

## Coxeter group, Coxeter system, Coxeter matrix

### Definition

- $M = (m_{ij})_{1 \leq i,j \leq n}$: a symmetric matrix: Coxeter matrix.
- $m_{ij} \in \mathbb{N} \cup \{\infty\}$ where $m_{ii} = 1$ and $m_{ij} > 1$ for $i \neq j$.
- $S = \{s_1, \ldots, s_n\}$: a generating set.
- $R = \{(s_i s_j)^{m_{ij}} = 1\}$: relations.
- $W(M) = \langle S | R \rangle$: Coxeter group of type $M$.
- $(W, S)$: a pair, called the Coxeter system of type $M$.

# The Coxeter-Dynkin diagrams

## Definition

The Coxeter-Dynkin diagram of Coxeter matrix $M$:

- A labeled graph.
- Nodes: $[n] = \{1, 2, \ldots, n\}$.
- Edges: node $i$ joined node $j$ by an edge labeled $m_{ij}$ if $m_{ij} \geq 3$. We often omit the label $3$.

# An example





## Example

- $M = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$, $S = \{a, b\}$,
- $W(M) = \left\{ a, b \middle| \, a^2 = b^2 = 1, (ab)^3 = 1 \right\} = \mathrm{Dih}_6 = S_3$.
- The Coxeter-Dynkin diagram of $M$ is: $\bullet\overset{3}{-}\bullet$.

# Reflection representation

## Definition

- $(W, S)$: a Coxeter system of type $M = (m_{ij})_{i,j \in [n]}$. $|S| = n$.
- $V$: a vector space of dimension $n$, with basis $e_1, \ldots, e_n$.
- $B(\square, \square)$: a bilinear form on $V$: $B(e_i, e_j) = -2\cos\frac{\pi}{m_{ij}}$. $B(e_i, e_j) = -2$ if $m_{ij} = \infty$.
- $Q(v) = \frac{1}{2}B(v, v)$: a quadratic form.
- $\rho_i(v) = v - B(v, e_i)e_i$: a linear transformation.
- $\rho\colon W \to \mathrm{GL}(V); r_1 r_2 \cdots r_q \mapsto \rho_1 \rho_2 \cdots \rho_q$ where $r_i \in S$.

# Properties



## Theorem

- $\rho_i$ is a reflection.
- $\rho_i$ preserves $B$: $B(\rho_i x, \rho_i y) = B(x, y)$.
- the order of $\rho_i \rho_j$ is $m_{ij}$.

# Finite reflection groups



## Definition

A finite reflection group is a finite linear group generated by reflections.

# The reflection representation is one-to-one

Consider the reflection representation $\rho\colon W \to GL(V)$.

- it is surjective.
- it is injective.

  Points:

  - prefundamental domain. (Definition 8, Theorem 9)
  - contragredient representation $\rho^*\colon W \to \mathrm{GL}(V^*)$. (Definition 10)
  - (Lemma 11, Theorem 12)

# Finite Coxeter group is a finite reflection group

Consider the map between two "bigger" "categories"
$\{$ all finite Coxeter groups $\} \to \{$ all finite reflection groups $\}$.

- it is indeed a map.
- it is injective.
- it is surjective.

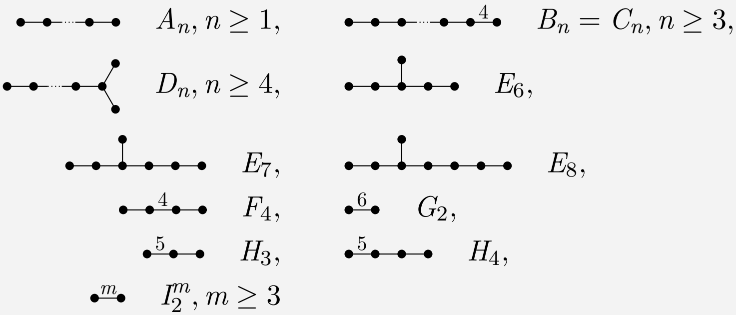
  Points:

  - irreducible representation and absolutely irreducible representation. (Definition 13, Theorem 14)
  - root system. (Definition 18)
  - positive definte symmetric bilinear form. (Lemma 20, Theorem 21)
  - then we have a Coxeter system for any finite reflection group. (Theorem 21)

# The main theorem

## Theorem

An irreducible Coxeter group is finite if and only if its Coxeter-Dynkin diagrams occurs in



## How to prove?

- **STEP 1.** " $\Longleftarrow$ ": $B$ is positive definite.
- **STEP 2.** connected diagrams.
- **STEP 3.** no circuit. use $Q(v) > 0$ for $v \neq 0$.
- **STEP 4.** exclude some infinite groups.
- **STEP 5.** use $Q$ to determinate that $4 > \sum_{k \neq i} B(e_i, e_k)^2$.
- **STEP 6.** most 3 edges from one node.
- **STEP 7.** if one node with 3 edges, these 3 edges are labeled 3.

## How to prove?

- **STEP 8.** if one edge labeled 6, most 2 nodes.
- **STEP 9.** if one node with 3 edges, all labeled 3.
- **STEP 10.** if one edge labeled 5, the two points of this edge either has no more edge, or has an edge labeled 3.
- **STEP 11.** most one node with 3 edges.
- **STEP 12.** exclude some subdiagrams.
- **STEP 13.** check all posible diagrams.

# The End

Many thanks to Prof. XXX.
Thank you for listening!